

# REDUCTIONS AND REAL FORMS OF HAMILTONIAN SYSTEMS RELATED TO $N$ -WAVE TYPE EQUATIONS

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**Abstract.** Reductions of  $N$ -wave type equations related to simple Lie algebras and the hierarchy of their Hamiltonian structures are studied. The reduction group  $G_R$  is realized as a subgroup of the Weyl group of the corresponding algebra. Some of the reduced equations are of physical interest.

**1. Preliminary.** The analysis and the classification of all reductions for the nonlinear evolution equations solvable by the inverse scattering method (ISM) is interesting and still open problem. We start with the well known form for the  $N$ -wave equations [1] :

$$i[J, Q_t] - i[I, Q_x] + [[I, Q], [J, Q]] = 0 \quad (1)$$

which is solvable by the ISM applied to the generalized system of Zakharov-Shabat type:

$$L(\lambda) = id_x + [J, Q(x, t)] - \lambda J, \quad J \in \mathfrak{h}. \quad (2)$$

The potential matrix

$$Q(x, t) = \sum_{\alpha \in \Delta_+} (q_\alpha(x, t)E_\alpha + p_\alpha(x, t)E_{-\alpha}) \in \mathfrak{g} \setminus \mathfrak{h} \quad (3)$$

takes values in the simple Lie algebra  $\mathfrak{g}$  with Cartan subalgebra  $\mathfrak{h}$ ,  $\Delta_+$  is the set of positive roots of  $\mathfrak{g}$ , and  $E_\alpha$ ,  $E_{-\alpha}$  and  $H_k$  form the Cartan-Weyl basis of  $\mathfrak{g}$ . Indeed the  $N$ -wave equation (1) is the compatibility condition  $[L(\lambda), M(\lambda)] = 0$ , where

$$M(\lambda) = id_t + [I, Q(x, t)] - \lambda I, \quad I \in \mathfrak{h} \quad (4)$$

**2. The reduction group.** Our basic tool is the reduction group  $G_R$  introduced by A. V. Mikhailov [2].  $G_R$  is a finite group which preserves the Lax representation (3), i.e. it ensures that the reduction constraints are automatically compatible with the evolution. Therefore  $G_R$  must have two realizations: 1)  $G_R \in \text{Aut } \mathfrak{g}$  and 2)  $G_R \in \text{Conf } \mathbb{C}$ . To each  $g_k \in G_R$  we relate a reduction condition for the Lax pair as follows [2]:

$$C_k(L(\Gamma_k(\lambda))) = L(\lambda), \quad C_k(M(\Gamma_k(\lambda))) = M(\lambda), \quad (5)$$

where  $C_k \in \text{Aut } \mathfrak{g}$  and  $\Gamma_k(\lambda) \in \text{Conf } \mathbb{C}$  are the images of  $g_k$ .

It is well known that  $\text{Aut } \mathfrak{g} = \mathfrak{V} \otimes \text{Aut}_0 \mathfrak{g}$  where  $\mathfrak{V}$  is the group of outer automorphisms (the symmetry group of the Dynkin diagram) and  $\text{Aut}_0 \mathfrak{g}$  is the group of inner automorphisms. We consider only those groups of inner automorphisms that preserve the form of  $L$  and  $M$ ; this means that  $G_R$  must preserve the Cartan subalgebra. Then

$G_R \in \mathfrak{V} \otimes \text{Ad}_H \otimes W(\mathfrak{g})$  where  $\text{Ad}_H$  is the group of similarity transformations with elements from the Cartan subgroup and  $W(\mathfrak{g})$  is the Weyl group of  $\mathfrak{g}$ . The reductions which lead to real forms of  $\mathfrak{g}$  may be realized by outer automorphisms of  $\mathfrak{g}$ :

$$C_p(X) = A_p \theta_p(X) A_p^{-1}, \quad \Gamma_p(\lambda) = \eta_p \lambda^*, \quad X \in \mathfrak{g}, \quad p = 1, 2, \quad (6)$$

where  $\theta_1(X) = X^\dagger$ ,  $\theta_2(X) = -X^*$ ,  $A_k \in G_R$  and  $\eta_1 = 1$ ,  $\eta_2 = -1$ . Of special interest is the possibility to embed  $G_R$  in the Weyl group  $W(\mathfrak{g})$  which can be done in a number of ways. Therefore it is important to distinguish between the nonequivalent reductions. In [3] the  $\mathbb{Z}_2$ -reduced  $N$ -wave systems for low-rank simple Lie algebras ( $\mathbf{A}_2$ ,  $\mathbf{C}_2$ ,  $\mathbf{G}_2$ ,  $\mathbf{A}_3$ ,  $\mathbf{B}_3$  and  $\mathbf{C}_3$ ) are described.

**3. Real forms as  $\mathbb{Z}_2$ -reductions.** It is well known that every real form  $\mathfrak{g}^\mathbb{R}$  can be extracted with an involutive Cartan automorphism  $\sigma$  (see e.g. [4]) and  $\theta(X) = X^\dagger$ ,  $X \in \mathfrak{g}$ . The automorphism that extracts a real form of the algebra  $\mathfrak{g}$  may be viewed as a reduction (6) with  $C_p = \sigma \theta_p$  and  $\Gamma(\lambda) = \eta_p \lambda^*$ . The compact real form  $\tilde{\mathfrak{g}}^\mathbb{R}$  of  $\mathfrak{g}$  corresponds  $\sigma = \text{Id}$ . The Cartan involution splits the root system of the real form  $\mathfrak{g}^\mathbb{R}$  into two subsystems of roots: 1) compact, if  $\sigma(E_\alpha) = E_\alpha$ ; and 2) noncompact, if  $\sigma(E_\alpha) = -E_\alpha$ .

Let  $\pi$  be the system of simple roots of the algebra and  $\pi_0$  be the set of compact simple roots. The Cartan involution which extracts the noncompact real form  $\mathfrak{g}^\mathbb{R}$  from the compact one is given by:

$$\sigma = \exp \left( \sum_{\alpha_k \in \pi \setminus \pi_0} \frac{2\pi i}{(\alpha_k, \alpha_k)} H_{\omega_k} \right) \in \text{Ad}_H. \quad (7)$$

Here  $H_{\omega_k}$  form basis in the Cartan subalgebra  $\mathfrak{h} \subset \mathfrak{g}$  dual to the fundamental weights  $\{\omega_k\}$ . In [3] one can find a number of examples for nontrivial reductions of the form (6) where  $A_k \in \text{Ad}_H \otimes W(\mathfrak{g})$ .

**4. Hamiltonian structures.** The interpretation of the ISM as a generalized Fourier transform allows one to prove that all  $N$ -wave type equations are Hamiltonian and possess a hierarchy of Hamiltonian structures [5]  $\{H^{(k)}, \Omega^{(k)}\}$ ,  $k = 0, \pm 1, \pm 2, \dots$ . The simplest Hamiltonian formulation of (1) is given by  $\{H^{(0)}, \Omega^{(0)}\}$  where  $H^{(0)} = \kappa_{0,p}(H_0 + H_{\text{int}})$  and:

$$H_0 = \frac{1}{2i} \int_{-\infty}^{\infty} dx \langle Q, [I, Q_x] \rangle, \quad H_{\text{int}} = \frac{1}{3} \int_{-\infty}^{\infty} dx \langle [J, Q], [Q, [I, Q]] \rangle, \quad (8)$$

$$\Omega^{(k)} = \frac{i\kappa_{k,p}}{2} \int_{-\infty}^{\infty} dx \left\langle [J, \delta Q(x, t)] \wedge \Lambda^k \delta Q(x, t) \right\rangle. \quad (9)$$

where  $\kappa_{k,1}/\kappa_{k,1}^* = -\eta_1^k$ ,  $\kappa_{k,2}/\kappa_{k,2}^* = (-1)^k$  and  $\langle \cdot, \cdot \rangle$  is the Killing form of  $\mathfrak{g}$  and  $\Lambda$  is the recursion operator, see [5]. Then the reduction conditions (6) require that both the symplectic forms and the Hamiltonians in the hierarchies are real:  $\Omega^{(k)} = (\Omega^{(k)})^*$  and  $H^{(k)} = (H^{(k)})^*$ .

One of the most efficient methods to analyze the Hamiltonian structures of the NLEE related to  $L$  (2) is based on the classical  $r$ -matrix method [6].  $r(\lambda)$  is defined by:

$$\{U(x, \lambda) \otimes U(y, \lambda)\} = [r(\lambda - \mu), U(x, \lambda) \otimes \mathbb{1} + \mathbb{1} \otimes U(y, \lambda)] \delta(x - y), \quad (10)$$

where  $U(x, \lambda) = [J, Q(x)] - \lambda J$ . If the Poisson brackets between the elements of  $U(x, \lambda)$  are the ones determined by  $\Omega^{(0)}$  then from (10) we get  $r(\lambda) = (R_0 + R_+ + R_-)/\lambda$  where:

$$R_0 = \sum_{j=1}^r H_j \otimes H_j^\vee, \quad R_\pm = \sum_{\alpha \in \Delta_\pm} \frac{E_{\pm\alpha} \otimes E_{\mp\alpha}}{\langle E_\alpha, E_{-\alpha} \rangle}, \quad (11)$$

and  $\langle H_j, H_k^\vee \rangle = \delta_{jk}$ . Next we integrate (10) taking special care of the zero boundary conditions for the potential  $Q(x)$  in (2). Thus for the Poisson brackets between the matrix elements of  $T(\lambda)$  – the scattering matrix of  $L$  corresponding to  $U(x, \lambda)$  we get:

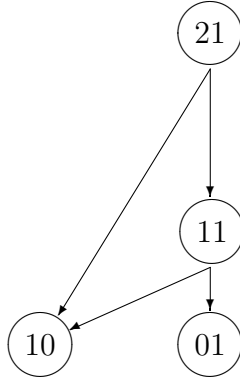
$$\{T(\lambda) \otimes T(\mu)\} = r_+(\lambda - \mu)T(\lambda) \otimes T(\mu) - T(\lambda) \otimes T(\mu)r_-(\lambda - \mu),$$

$$r_\pm(\lambda) = \frac{R_0}{\lambda} \mp i\pi\delta(\lambda)(R_+ - R_-). \quad (12)$$

Note that  $R_0$  and  $\pm(R_+ \pm R_-)$  are invariant with respect to  $G_R$ . As a result both (10) and (12) survive the reductions for which  $\Omega^{(0)}$  remains non-degenerate, such as e.g. reductions extracting the real forms of  $\mathfrak{g}$ .

Degeneracies appear only in the case when the reduction automorphism of the algebra maps  $C(J) = -J$  and  $\Gamma(\lambda) = -\lambda$ ; note that no complex conjugation is involved here. Then both the canonical 2-form  $\Omega^{(0)}$  and the  $r$ -matrix become degenerate, so the corresponding  $N$ -wave equations do not allow Hamiltonian formulation with canonical Poisson brackets; however they still possess a hierarchy of Hamiltonian structures  $\Omega^{(2k+1)}$  being nondegenerate.

**5. Example.** *The  $N$ -wave systems for  $\mathbf{C}_2 \simeq sp(4)$  algebra.*



**Figure 1:** This is the wave-decay diagram for the  $sp(4)$  algebra. To each positive root of the algebra  $\underline{k}\mathbf{n} \equiv k\alpha_1 + n\alpha_2$  we put in correspondence a wave of type  $\underline{k}\mathbf{n}$ . If the positive root  $\underline{k}\mathbf{n} = \underline{k}'\mathbf{n}' + \underline{k}''\mathbf{n}''$  can be represented as a sum of two other positive roots, we say that the wave  $\underline{k}\mathbf{n}$  decays into the waves  $\underline{k}'\mathbf{n}'$  and  $\underline{k}''\mathbf{n}''$  as shown on the diagram to the left.

**A)** After a reduction of hermitian type (6) with  $A_1 = \text{diag}(s_1, s_2, 1/s_2, 1/s_1)$  and  $\eta_1 = \pm 1$  we obtain  $a_i = \eta_1 a_i^*$ ,  $b_i = \eta_1 b_i^*$ , and

$$p_{10} = -\eta_1 s_1 / s_2 q_{10}^*, \quad p_1 = -\eta_1 s_2^2 q_1^*, \quad p_{11} = -\eta_1 s_1 s_2 q_{11}^*, \quad p_{21} = -\eta_1 s_1^2 q_{21}^*. \quad (13)$$

The corresponding 4-wave system is described by the following interaction Hamiltonian:

$$H_{\text{int}} = 4\kappa \int_{-\infty}^{\infty} (s_1 s_2 (q_{11} q_1^* q_{10}^* + \eta_1 q_{11}^* q_1 q_{10}) - s_1^2 (q_{21} q_{11}^* q_{10}^* + \eta_1 q_{21}^* q_{11} q_{10})) \, dt. \quad (14)$$

where  $\kappa = a_1 b_2 - a_2 b_1$ . The case  $\eta_1 = 1$  and  $s_1 = s_2 = 1$  leads to the (compact) real form  $sp(4, 0)$  of  $\mathbf{C}_2$ -algebra and it is equivalent to the 4-wave interaction, see [1].

Physically we can assign to each root  $\alpha$  an wave with wave number  $k_\alpha$  and frequency  $\omega_\alpha$ . Each of the elementary decays  $\alpha_i = \alpha_j + \alpha_k$  is possible if  $k_{\alpha_i} = k_{\alpha_j} + k_{\alpha_k}$  and  $\omega(k_{\alpha_i}) = \omega(k_{\alpha_j}) + \omega(k_{\alpha_k})$ . This can also be written using so-called wave-decay diagrams [1]; an example of such diagrams is shown in fig.1.

**B)** For  $\eta = -1$  and  $s_1 = s_2 = 1$  if we identify  $q_{10} = Q$ ,  $q_{11} = E_p$ ,  $q_{21} = E_a$  and  $q_1 = E_s$ , where  $Q$  is the normalized effective polarization of the medium and  $E_p$ ,  $E_s$  and  $E_a$  are the normalized pump, Stockes and anti-Stockes wave amplitudes respectively, then we obtain the system of equations studied, e.g. in [7]. This approach allowed us to derive a new Lax pair for (14).

**6. Conclusions.** The  $\mathbb{Z}_2$ -reductions which act on  $\lambda$  by  $\Gamma(\lambda) = \lambda^*$  may be viewed as Cartan involution and lead to restricting of the system to a specific real form of the algebra  $\mathfrak{g}$ . They preserve the canonical symplectic form  $\Omega^{(0)}$  and the classical  $r$ -matrix.

The results can be extended in several directions: i) for more general reduction groups  $G_R$ ; ii) for NLEE with other dispersion laws. This would allow us to study the reductions of multicomponent NLS-type equations, two-dimensional Toda type systems etc.; iii) for Lax operators with more complicated  $\lambda$ -dependence. This would allow us to investigate also more complicated reduction groups as e.g.  $\mathbb{T}$ ,  $\mathbb{O}$  and the possibilities to imbed them as subgroups of the Weyl group of  $\mathfrak{g}$ .

**Acknowledgement.** The authors thank Dr. N. A. Kostov for numerous useful discussions.

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